Econ 201C: Problem Set 2
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1. **Risk Sharing in Competitive Market**

   A competitive market of risk-neutral firms competes to employ a risk-averse worker over time, $t \in \{1, 2, \dots T\}$. A firm can commit to its contract, but the worker cannot. A worker's common productivity θ is initially unknown. Every period firms and the worker learn symmetrically about her productivity via output $y_t \sim f(\cdot \mid \theta)$, where y^t is the history of output up to and including time t.

   Each period t runs as follows:

   (1) The perfectly competitive market of firms makes contract offers $\{w_s(y^{s-1})\}_{s \geq t}$.

   (2) The worker choose her favorite contract.

   (3) Output $y_t \sim f(\cdot \mid \theta)$ is publicly realized. This is IID conditional on θ.

   (4) The agent gets utility $u(w)$, where $u(\cdot)$ is increasing and strictly concave. His employer gets $\pi = y_t - w_t$.

   Given common discount rate δ, denote continuation values by

   $$U_t(y^{t-1}) = \sum_{s \geq t} \delta^{s-t} E_t[u(w_s)]$$

   and

   $$\Pi_t(y^{t-1}) = \sum_{s \geq t} \delta^{s-t} E_t[y_s - w_s],$$

   where the expectation E_t is taken at the start of period t, given y^{t-1}.[1]

   (a) We wish to find which contract the competitive market offers. Argue that we can restrict ourselvelves to contracts in which (i) the agent never switches firms, and (ii) the firm's continuation profits are never strictly positive.

   (i) Firms are identical and risk-neutral, and the worker doesn't care about the identity of the employer, only the continuation utilities. If there's some equilibrium where the worker switches firms, we can simply write a contract where the original firm promises the wage stream the worker would've received from switching, and the total firm profits are the same.

   (ii) The firm's continuation profit is always non-positive via Bertrand-like competition from other firms. If a firms continuation profit $\Pi_t(y^{t-1}) > 0$ for some t, another firm could offer a contract raising future wages by a present value amount ε, and secure continuation profit $\Pi_t(y^{t-1}) - \varepsilon \geq 0$ when the worker switches to their contract.

[1] Observe that period t's wage is chosen, and the agent chooses his contract before y_t is observed. Thus, in a one-period problem, we can obtain full insurance. This constrasts with the mode in class where the agent can quit after the output is revealed.

Competition between the firms will drive initial profits Π_1 to zero. We thus wish to solve

$$\max_{w_t(y^{t-1})} \sum_{t=1}^{T} \delta^{t-1} E_1 \left[u(w_t(y^{t-1}))\right]$$

$$\text{s.t.} \quad (\text{IR}_t) \quad \Pi_t(y^{t-1}) \leq 0 \quad \forall t$$
$$(\text{IR}_1) \quad \Pi_1 = 0$$

(b) Suppose $T = 2$ and the discount rate is $\delta = 1$. Productivity is $\theta \in \{0, 2\}$ with equal probability. Output is perfectly revealing, $y_t = \theta$. What is the equilibrium contract?

A contract consists of $(w_1, w_2(0), w_2(2))$, where the argument in $w_2(\cdot)$ is the output $y_1 = \theta$ observed. In period 2, we have

$$\Pi_2(\theta) = \theta - w_2(\theta),$$

so using (IR_t), $w_2(0) \geq 0$, $w_2(2) \geq 2$.

Using (IR_1), we also have

$$\Pi_1 = 0 = \mathbb{E}[y_1 + y_2] - (w_1 + (0.5)w_2(0) + (0.5)w_2(2))$$
$$\implies 2 = w_1 + (0.5)w_2(0) + (0.5)w_2(2).$$

Since worker is risk-averse, u is concave, and the objective function is maximized at $w_1 = w_2(0) = w_2(2) = 1$ (smoothed consumption/Jensen's inequality). But by (IR_1), we must set the lowest $w_2(2) = 2$. This yields

$$1 = w_1 + (0.5)w_2(0),$$

and so smoothing consumption again, we have $w_1 = w_2(0) = 2/3$. In this agent-optimal contract, the worker gets partial insurance, in that if $\theta = 0$, worker is still paid $2/3$, but once the high productivity is revealed, competition forces wages up to 2, and the worker captures output.

(c) Suppose $T = \infty$, and the discount rate is $\delta \in (0, 1)$. Productivity is $\theta \in \{0, 2\}$ with equal probability. If $\theta = 0$ then the agent produces output $y_t = 0$ each period. If $\theta = 2$ then the agent produces $y_t = 4$ with probability $1/2$ and $y_t = 0$ with probability $1/2$. What is the equilibrium contract?

We have two states: 1) no 4 has been observed up to t, and 2) a 4 has been observed. By proposition/consumption-smoothing logic, we know that the optimal contract takes the "minimum-wage" form

$$w_t(y^{t-1}) = \begin{cases} w_L & : \quad \text{State (1)} \\ w_H & : \quad \text{State (2).} \end{cases}$$

In state (2), firms realize productivity $\theta = 2$, and thus $\mathbb{E}[y_t] = 2$. Plugging this into (IR_t), we have

$$\sum_{s \geq t} \delta^{s-t}(2 - w_H) \leq 0$$

$$\implies \frac{2-w_H}{1-\delta} \leq 0 \implies w_H \geq 2,$$

so firms compete up the post-revelation wage to $w_H = 2$.

Now, we solve for w_L. If $\theta = 0$, then we are in State (1) forever, and wages are $\frac{w_L}{1-\delta}$. If $\theta = 2$, then the probability that some round t' is the first round where output 4 is observed is $\left(\frac{1}{2}\right)^{t'}$. The present value of wages would then be:

$$w_L(1+\delta+\ldots+\delta^{t'-1} + 2(\delta^{t'}+\delta^{t'+1}) = w_L\left(\frac{1-\delta^{t'}}{1-\delta}\right) + 2\left(\frac{\delta^{t'}}{1-\delta}\right).$$

Taking expectations over t', we have

$$E[\delta^{t'}] = \sum_{t'=1}^{\infty}\left(\frac{1}{2}\right)^{t'}\delta^{t'} = \frac{\delta}{2-\delta},$$

so the expected present value of wages under $\theta = 2$ is

$$w_L\left(\frac{2}{2-\delta}\right) + 2\left(\frac{\delta}{(1-\delta)(2-\delta)}\right).$$

Using (IR_1) and knowing $\theta = 0, 2$ with 50/50 probability, we equate the expected present value of output and wages:

$$\frac{1}{2}\left(w_L\left(\frac{2}{2-\delta}\right) + 2\left(\frac{\delta}{(1-\delta)(2-\delta)}\right)\right) + \frac{1}{2}\left(\frac{w_L}{1-\delta}\right) = \frac{1}{2}(0) + \frac{1}{2}\left(\frac{2}{1-\delta}\right) = \frac{1}{1-\delta}.$$

Solving, we find

$$w_L = \frac{4(1-\delta)}{4-3\delta}.$$

2. **Risk Sharing with Hidden Wage Offers**

A risk neutral firm employs a risk-averse worker. Time is discrete and infinite, $t \in \{1, 2, \ldots\}$. Both players have discount rate $\delta \in (0, 1)$. The worker has constant productivity q at the firm and receives IID private wage offers x each period. Each period, the worker can then stick with the contract, or quit the firm and take the outside offer.

More precisely, the firm commits to a deterministic sequence of wages $\{w_t\}$.[2] At the start of the period, the worker privately sees the outside offer, x. This offer has a strictly positive density $f(\cdot)$, distribution $F(\cdot)$ and support $[0, 1]$. If the worker takes the outside offer she gets $u(x)$ forever (i.e. she gets total $u(x)/(1-\delta)$), while the firm gets zero. If the worker rejects the outside offer she gets $u(w)$ this period, while the firm gets $\pi = q - w$, where $q > 1$.

We wish to find the firm's optimal contract. In particular, we wish to write down the firm's problem recursively. Let U be the worker's continuation value at the start of the current period, before x is realized, including the utility the agent obtains when she leaves the firm. Let U_+ be the continuation utility promised tomorrow, and let $\Pi(U)$ be the firm's profit function.

[2] We do not allow the agent to report her outside offer each period. I suspect this is without loss, but have not proved it.

(a) The worker quits if her outside wage offer exceeds a threshold, z. How is z determined?

The worker's present value from quitting at any round when offered x is $\frac{u(x)}{1-\delta}$. If she stays, then she gets current wage and δ times promised continuation utility. z is determined by the indifference condition:

$$\frac{u(z)}{1-\delta} = u(w) + \delta U_+ \iff z = u^{-1}\left((1-\delta)(u(w)+\delta U_+\right).$$

(b) Write down the firm's profit maximization problem recursively. That is, maximize $\Pi(U)$ subject to promise keeping and z being determined as in part (a).

The firm's state variable is promised worker utility U, and they can choose w, U_+, z. For any given z, worker stays if $x \leq z$, which happens with $F(z)$ probability. Thus, the workers objective function is

$$\Pi(U) = \max_{w,U_+,z} F(z)\left(q - w + \delta\Pi(U_+)\right).$$

For promise-keeping, U must equal the worker's promised utility, i.e.

$$U = F(z)\left(u(w) + \delta U_+\right) + \int_z^1 \frac{u(x)}{1-\delta} f(x)dx.$$

Finally, we impose the cutoff constraint:

$$\frac{u(z)}{1-\delta} = u(w) + \delta U_+.$$

(c) Assume that $\Pi(U)$ is decreasing and concave in U over the relevant range.[3] Write down the Lagrangian corresponding to the firm's problem. Show that the optimal choices of z, w and U_+ are related by the equation

$$\frac{1}{u'(w)} = -\Pi'(U_+) = \frac{f(z)}{F(z)^2}\frac{\Pi(U)(1-\delta)}{u'(z)} - \Pi'(U).$$

What is the interpretation of this?

We have

$$\begin{aligned}\mathcal{L} &= F(z)\left(q - w + \delta\Pi(U_+)\right) \\ &\quad + \lambda_1\left[F(z)\left(u(w)+\delta U_+\right) + \int_z^1 \frac{u(x)}{1-\delta}f(x)dx - U\right] + \lambda_2\left[(1-\delta)(u(w)+\delta U_+) - u(z)\right].\end{aligned}$$

FOC w.r.t. w:

$$-F(z) + \lambda_1 F(z)u'(w) + \lambda_2(1-\delta)u'(w) = 0$$

[3] $\Pi(U)$ is clearly weakly decreasing (why?). However it may not be concave under a deterministic wage scheme, e.g. if there is an at atom in $F(\cdot)$. One can restore concavity but at the cost of having a random mechanism. We ignore these issues here.

$$\implies \frac{1}{u'(w)} = \lambda_1 + \frac{\lambda_2(1-\delta)}{F(z)}.$$

FOC w.r.t. U_+:

$$F(z)\delta\Pi'(U_+) + \lambda_1 F(z)\delta + \lambda_2(1-\delta)\delta = 0$$

$$\implies -\Pi'(U_+) = \lambda_1 + \frac{\lambda_2(1-\delta)}{F(z)}.$$

Combining this with the FOC for w yields the first equality. Finally, the FOC w.r.t. z:

$$f(z)\left(q - w + \delta\Pi(U_+)\right) + \lambda_1\left[f(z)(u(w) + \delta U_+) - \frac{u(z)}{1-\delta}f(z)\right] - \lambda_2 u'(z) = 0,$$

By the cutoff condition, the bracketed term is 0. Furthermore, since $\Pi(U)$ equals $F(z)\left(q - w + \delta\Pi(U_+)\right)$ at the optimized values, we have that

$$\lambda_2 = \frac{f(z)\left(q - w + \delta\Pi(U_+)\right)}{u'(z)} = \frac{f(z)\Pi(U)}{F(z)u'(z)}.$$

Furthermore, by the envelope condition:

$$\Pi'(U) = \frac{\partial\mathcal{L}}{\partial U} = -\lambda_1.$$

Substituting λ_1, λ_2 into the equality yields the final equality:

$$\frac{1}{u'(w)} = \frac{f(z)}{F(z)^2}\Pi(U)\frac{(1-\delta)}{u'(z)} - \Pi'(U).$$

The first equality equates the marginal wage cost of giving the worker utility today $(1/u'(w))$ with the marginal profit cost of promising more utility tomorrow $-\Pi'(U)$. These are also equated to the marginal cost of increasing the worker's promised utility today $-\Pi'(U)$, plus the quitting term $\frac{f(z)}{F(z)^2}\Pi(U)\frac{(1-\delta)}{u'(z)}$, since raising U also increases the cutoff z, making the worker less likely to quit, which is desirable for the firm.

(d) Argue that U, w and z increase over time so long as the worker has some probability of quitting, $z < 1$. What is the intuition for this?

Using

$$-\Pi'(U_+) = \frac{f(z)}{F(z)^2}\frac{\Pi(U)(1-\delta)}{u'(z)} - \Pi'(U),$$

we see that when $z < 1$, $f(z) > 0$, as well as $F(z), u'(z), \Pi(U) > 0$. Thus, the quitting term is positive, and we have

$$-\Pi'(U_+) > -\Pi'(U).$$

Since $\Pi(U)$ is concave, $-\Pi'(U)$ is increasing in U, so $U_+ > U$, so promised utility rises over time as the workers stays.

Using

$$\frac{1}{u'(w)} = -\Pi'(U_+),$$

since U_+ is increasing, $-\Pi'(U_+)$ is increasing, so $u'(w)$ is decreasing. Since $u(w)$ is concave, w is also rising over time.

Finally, using the cutoff condition:

$$z = u^{-1}\left((1-\delta)(u(w) + \delta U_+\right),$$

since w and U_+ are rising, z is rising as well.

Intuitively, if the worker has not quit, the firm gradually rewards the worker by promising more utility/wages, which in turn make the worker want to leave less. The contract becomes more valuable over time as the worker gets attached to the firm.

3. **Selling Concessions**

A stadium (the "principal") is selling a concession to an agent to sell hot dogs at a baseball game. The agent has private information about consumers' demand for hot dogs, $\theta \sim F[\underline{\theta}, \overline{\theta}]$. Demand is linear with $p = \theta - q$, where q is the quanity sold. The agent's cost is zero, so his utility is

$$u = q(\theta - q) - t$$

where t is the transfer paid to the principal. His outside option is $\bar{u} = 0$. The principal's profit is $\pi = t$. Assume $\frac{1-F(\theta)}{f(\theta)}$ is decreasing in θ.

Consider the first-best problem. Suppose that the principal can observe the agent's type θ and fully extract his rents.

(a) What is the first-best allocation $q(\theta)$?

Total welfare is given by

$$W = q(\theta = q).$$

Maximizing this w.r.t. q yields

$$q^{FB}(\theta) = \theta/2.$$

(b) Suppose $\theta \sim U[0, 1]$. Calculate the principal's expected profit.

The principal makes the agent's IR bind by choosing transfer t so that agent's utility is 0. At q^{FB},

$$t = \theta/2(\theta - \theta/2) = \theta^2/4.$$

Taking the expectation over θ yields:

$$1/4E[\theta^2] = 1/4\int_0^1 \theta^2 d\theta = 1/12.$$

For the rest of the question, suppose θ is only known by the agent. Suppose that the quantity of hot dogs sold q is contractible. A mechanism $\langle q(\tilde{\theta}), t(\tilde{\theta})\rangle$ consists of quantity $q(\tilde{\theta}) \in \mathbb{R}_+$ and transfer, $t(\tilde{\theta}) \in \mathbb{R}$, where $\tilde{\theta}$ is the agent's reports type.

(c) Write down the agent's expected utility in an IC mechanism.

Let $U(\theta) = q(\theta)(\theta - q(\theta)) - t(\theta)$ be agent's truth-telling utility, and $u(\theta, \widetilde{\theta}) = q(\widetilde{\theta})(\theta - q(\widetilde{\theta})) - t(\widetilde{\theta})$ be agent of type θ's utility pretending to be $\widetilde{\theta}$. In an IC mechanism, truth-telling is optimal, i.e.

$$U(\theta) = \max_{\widetilde{\theta}} u(\theta, \widetilde{\theta}).$$

Using envelope theorem, we get

$$U'(\theta) = q(\theta),$$

and with the FTC, we find

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)ds.$$

Note the monotonicity condition that $q(\theta)$ should be increasing.

(d) What is the principal's profit maximizing mechanism?

We solve the relaxed problem, ignoring monotonicity. First observe that profits/expected transfers equal surplus minus utility, i.e.

$$\Pi = E[\theta q(\theta) - q(\theta)^2 - U(\theta)].$$

Second, we calculate using IBP and ICFOC from part (c):

$$E[U(\theta)] = U(\underline{\theta}) + E\left[q(\theta)\frac{1 - F(\theta)}{f(\theta)}\right].$$

Substituting this into the profit equation and using the fact that $\underline{\theta}$'s IR binds in the optimal contract, we get

$$\Pi = E\left[\left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) q(\theta) - q(\theta)^2\right].$$

Define $MR(\theta) := \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)$. Assuming that $\frac{1-F(\theta)}{f(\theta)}$ is decreasing means $MR(\theta)$ is increasing, which satisfies monotonicity, and thus, incentive compatibility. Pointwise differentiating w.r.t. q, we get FOC

$$q(\theta) = \frac{MR(\theta)}{2},$$

and so the optimal allocation is $q^\star(\theta) = \left(\frac{MR(\theta)}{2}\right)_+$. Plugging back in for optimal transfers:

$$t^\star(\theta) = \theta q^\star(\theta) - q^\star(\theta)^2 - \int_{\underline{\theta}}^{\theta} q^\star(s)ds.$$

(e) Suppose $\theta \sim U[0,1]$. Calculate the principal's expected profit. Show that it is less than the profit in (b).

We have that $MR(\theta) = 2\theta - 1$, so $q^\star(\theta) = (\theta - 1/2)_+$. Thus, for $\theta \leq 1/2$, profit is zero, and for $\theta > 1/2$, we have

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) q(\theta) - q(\theta)^2 = (2\theta-1)(\theta-1/2) - (\theta-1/2)^2 = (\theta-1/2)^2.$$

Thus, taking the expectation yields

$$\int_{1/2}^{1} (\theta - 1/2)^2 d\theta = 1/24,$$

which is less than the first-best expected profit of 1/12. The principal must give up information rents to the high-demand agents, giving higher quantities to them, while lower types get no allocation at all.

Now, suppose that the quantity q is not contractible, so once the principal sells the concesession to the agent, the agent can sell however much they like. A mechanism $\langle y(\tilde{\theta}), t(\tilde{\theta})\rangle$ consists of an allocation probability $y(\tilde{\theta}) \in [0,1]$ and transfer $t(\tilde{\theta}) \in \mathbb{R}$.

(f) Fix $\langle y, t\rangle$. How will the agent choose the quantity sold, q, if they buy the concession?

Since the quantity q is not contractible and the transfer is a sunk cost, after the agent buys the concession, they will simply sell the monopoly quantity, i.e. $q^\star(\theta) = \theta/2$.

(g) What is the principal's profit maximizing mechanism?

If agent of type θ buys, his total value is $\theta/2(\theta - \theta/2) = \theta^2/4$. Let $U(\theta) = y(\theta)(\theta^2/4) - t(\theta)$ be agent's truth-telling utility, and let $u(\theta, \widetilde{\theta}) = y(\widetilde{\theta})(\theta^2/4) - t(\widetilde{\theta})$ be the utility from pretending to be type $\widetilde{\theta}$. In an IC mechanism,

$$U(\theta) = \max_{\widetilde{\theta}} u(\theta, \widetilde{\theta}),$$

and using envelope theorem, we see

$$U'(\theta) = y(\theta)(\theta/2).$$

Thus, by FTC, we have

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} y(s)(s/2)ds.$$

Now assuming IC, the principal maximizes profits/transfers, which equal total surplus minus agent utility:

$$\Pi = E\left[y(\theta)(\theta^2/4) - U(\theta)\right].$$

Calculating the agent's expected utility as before with IBP, ICFOC, and $U(\underline{\theta}) = 0$, we get:

$$E[U(\theta)] = \left[y(\theta)(\theta/2)\frac{1-F(\theta)}{f(\theta)}\right],$$

and profit is

$$\Pi = E\left[y(\theta)\left((\theta^2/4) - (\theta/2)\frac{1-F(\theta)}{f(\theta)}\right)\right].$$

Thus, the optimal mechanism is bang-bang, where $y^\star(\theta)$ is an indicator for when the term in the parenthesis is nonnegative, i.e.

$$y^\star(\theta) = \mathbb{I}\left[\theta \geq 2\frac{1-F(\theta)}{f(\theta)}\right].$$

Thus, we can define a cutoff rule for the principal $y^\star(\theta) = \mathbb{I}[\theta \geq \theta^\star]$, where $\theta^\star$ is defined via the fixed point $\theta^\star = 2\frac{1-F(\theta^\star)}{f(\theta^\star)}$. Note since $\frac{1-F(\theta)}{f(\theta)}$ is decreasing, $y^\star$ is monotone, and so the mechanism is IC. Optimal transfers are thus given by:

$$t^\star(\theta) = y^\star(\theta)(\theta^2/4) - \int_{\underline{\theta}}^{\theta} y^\star(s)(s/2)ds,$$

which is 0 when $\theta < \theta^\star$, and $(\theta^\star)^2/4$ when $\theta \geq \theta^\star$.

(h) Suppose $\theta \sim U[0,1]$. Calculate the principal's expected profit. Show that it is less than the profit in (e).

We solve

$$\theta^\star = 2(1-\theta^\star) \implies \theta^\star = 2/3.$$

Thus, expected transfers/profits equal

$$0 \cdot P(\theta < 2/3) + (2/3)^2/4 \cdot P(\theta \geq 2/3) = 1/27,$$

which is less than that in part (e). When q is not contractible, the principal can no longer screen types by assigning quantities, and instead uses a cutoff rule to decide whether to sell a concession.

4. **Pricing with Two Customer Groups**

A firm with marginal cost c faces agents with private values who want one unit of a good. There is mass $1/5$ of business customers with values $\theta \sim U[1,2]$, and mass $4/5$ leisure customers with values $\theta \sim U[0,1]$. The firm does not know if a customer is "business" or "leisure". In a direct revelation mechanism $\langle q(\theta), t(\theta)\rangle$, an agent who reports type $\tilde{\theta}$ gets quantity $q(\tilde{\theta}) \in [0,1]$ and pays $t(\tilde{\theta})$. Agent θ then gets utility $u = \theta q(\tilde{\theta}) - t(\tilde{\theta})$, and the firm receives profit $\pi = t(\tilde{\theta}) - cq(\tilde{\theta})$.[4]

We showed in class that the firm chooses the mechanism to solve

$$\max_{\langle q,t\rangle} \Pi = E\left[(MR(\theta) - c)q(\theta)\right]$$

$$\text{s.t.} \quad q(\theta) \text{ increasing in } \theta$$

where $MR(\theta) := \theta - \frac{1-F(\theta)}{f(\theta)}$.

[4] We assume agents just report their value and not whether they are "business" or "leisure". This is without loss.

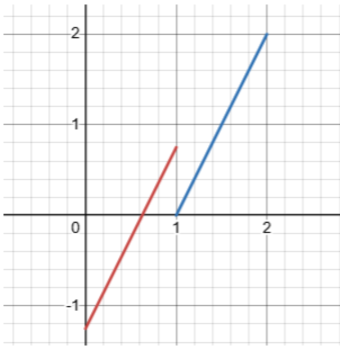

Figure 1: $MR(\theta)$

(a) Plot $MR(\theta)$ as a function of θ.

For $\theta \in [0, 1]$, we are in the 4/5 mass of leisure customers, so $F(\theta) = \frac{4}{5}\theta$, $f(\theta) = \frac{4}{5}$, and so

$$MR(\theta) = 2\theta - \frac{5}{4}.$$

For $\theta \in [1, 2]$, we are in the upper 1/5 mass of business customers, so $F(\theta) = 4/5 + \frac{1}{5}(\theta - 1)$, $f(\theta) = \frac{1}{5}$, and so

$$MR(\theta) = 2\theta - 2.$$

Plotting this (ignoring the limit at $\theta = 1$), we have Figure 1.

(b) Suppose the firm has marginal cost $c = 0$. What is the profit-maximizing allocation? How can the firm implement this via prices?

$q^\star(\theta)$ is bang-bang for when $MR(\theta) \geq 0$, i.e. $q^\star(\theta) = \mathbb{I}[\theta \geq 5/8]$, which is indeed a monotonic allocation. The firm can implement with a constant price $p = 5/8$, so only types $\theta \geq 5/8$ will purchase and get $u \geq 0$.

(c) Suppose the firm has marginal cost $c = 1$. What is the profit-maximizing allocation? How can the firm implement this via prices?

$q^\star(\theta) = \mathbb{I}[MR(\theta) \geq 1] = \mathbb{I}[\theta \geq 3/2]$, i.e., the firm never sells to leisure customers. Again, the firm can implement this with a single price $p = 3/2$, so only business customers with $\theta \geq 3/2$ will purchase and capture nonnegative utility.

(d) Suppose the firm has marginal cost $c = 1/2$. What is the profit-maximizing allocation? How can the firm implement this via prices?

$MR(\theta)$ has a jump when valued at 1/2, so in order to get a monotonic allocation, we iron it out and look at the endpoints. Setting $MR(\theta) = 1/2$, we have candidates $\theta \geq 7/8 = p_1$, which yields profit

$$(7/8 - 1/2) \cdot P(\theta \geq 7/8) = (3/8)((4/5)(1 - 7/8) + 1/5) = 9/80$$

and $\theta \geq 5/4 = p_2$, which yields profit

$$(5/4 - 1/2) \cdot P(\theta \geq 5/4) = (3/4)((1/5)(2 - 5/4) = 9/80$$

as well. So the firm is indifferent between $q^\star = \mathbb{I}[\theta \geq 7/8]$, $p = 7/8$ and $q^\star = \mathbb{I}[\theta \geq 5/4]$, $p = 5/4$. The firm must tradeoff selling at a low price to capture demand from low leisure types and charging a higher price to just high business types. At $c = 1/2$, they are indifferent between these two options.

(e) Suppose the firm has mass $1/5$ of the good to sell, and marginal cost $c = 0$ up to this capacity. What is the profit-maximizing allocation? How can the firm implement this via prices?

With $1/5$ mass, the firm would like to allocate quantity to types with the highest marginal revenues (high business types and high leisure types), but this requires ironing $MR(\theta)$ on some interval across $\theta = 1$ to achieve a monotone allocation rule. The firm maximizes $E[MR(\theta)q(\theta)]$ subject to the additional constraint that the total allocation across all types is $1/5$, i.e. $E[q(\theta)] \leq 1/5$. Attaching a Langrange multiplier to this constraint, the firm solves the relaxed problem of maximizing

$$E[(MR(\theta) - \lambda)q(\theta)],$$

so firm allocates where $MR(\theta) \geq \lambda$. Suppose the ironed interval is $a < 1 < b$, where at the endpoints, $MR(a) = MR(b) = \lambda$. Solving, we get

$$2a - 5/4 = \lambda = 2b - 2 \implies a = \lambda/2 + 5/8,\ b = \lambda/2 + 1.$$

By the ironing condition, the average marginal revenue across $[a, b]$ must equal λ:

$$\int_a^b (MR(\theta) - \lambda)f(\theta)d\theta = 0,$$

and we know from part (c) that this is true when $c = \lambda = 1/2$, and so $[a, b] = [7/8, 5/4]$. The firm thus sells to business customers with types $\theta \geq 5/4$, which have mass $(1/5)(2 - 5/4) = 3/20$. The remaining $1/20$ gets allocated across the types within the ironed interval, which has mass $(4/5)(1 - 7/8) + (1/5)(5/4 - 1) = 3/20$, so we can meet our capacity by randomly allocating to $1/3$ of types in the ironed interval. Thus, the optimal monotonic allocation is

$$q^\star(\theta) = \begin{cases} 0 & : \quad \theta < 7/8 \\ 1/3 & : \quad \theta \in [7/8, 5/4) \\ 1 & : \quad \theta \geq 5/4. \end{cases}$$

Using $E[t(\theta)] = E[\theta q(\theta) - \int_0^\theta q(s)ds]$, the corresponding optimal transfer payments are

$$E[t^\star(\theta)] = \begin{cases} 0 & : \quad \theta < 7/8 \\ 7/24 & : \quad \theta \in [7/8, 5/4) \\ 9/8 & : \quad \theta \geq 5/4. \end{cases}$$

Thus, the firm can implement this allocation with prices by charging $p = 9/8$ to receive the good for sure, and $p = 7/24$ to receive the good with $1/3$ probability. Indeed,

$$\theta - 9/8 \geq (1/3)\theta - 7/24 \iff \theta \geq 5/4,$$

so high business types select into the guaranteed option, and

$$(1/3)\theta - 7/24 \geq 0 \iff \theta \geq 7/8,$$

so types in the ironed interval select into the lottery option, while low leisure types will not be supplied. This allocation schedule is monotonic, and thus, incentive compatible.